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STRANGE ATTRACTORS IN PIECEWISE-LINEAR VECTOR FIELDS

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Abstract: This note provieds an example of piecewise-linear vector field whose Poincare map converges to a one-dimensional map.

§ 0. Introduction.

Rössler's spiral attractor is a typical strange attractor in a 3-dimensional vector fields defined by a concrete differential equation [1]. Under the Poincaré map of the vector field, a rectangl region is contracted vertically, expanded horizontally, bended and mapped into itself. Since the vertical contraction is very strong, the bifurcation of the Poincaré map is often studied by reducing to a 1-dimensional map. Whether we can reduce the behavior of 2-dimensional map to that of 1-dimensional map or cannot is a matter for argument. This problem is studied by many people, including M.Yuri[2] and M.Misiurewicz[3]. However, there is no example which shows that a Poincaré map defined from a concrete ordinary differential equation can be reduced to some 1-dimensional map. Rössler has shown that, as same as the case of smooth nonlinear term, the spiral attractors appear in ordinary differential equations with piecewise-linear functions as nonlinear terms [4].

This note provieds an example of piecewise-linear vector field whose Poincaré map converges to a 1-dimensional map. More precisely, when suitable parameter of the vector field goes to a limit, we will show that

1) the Poincaré map converges to a 1-dimensional map;

2) the properties of periodic points correspond between 2-dimensional map and 1-dimensional map;

3) there exists a limit of the vector field which can define a semi-flow on Rössler's paper sheet model.

All proofs of theorems will be given in [7].

§ 1 Piecewise-Linear Vector Fields.

Definition 1. Define a piecewise-linear vector field $\xi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$\xi(x) = \begin{cases} Mx & \text{if } \langle a,x\rangle \leq 1 \\ M_1 x - p & \text{if } \langle a,x\rangle > 1 \end{cases}$$

where $a = \begin{pmatrix}1\\0\\1\end{pmatrix}$, $x,p \in \mathbb{R}^3$, $M = \begin{bmatrix}\sigma & -\omega & 0\\ \omega & \sigma & 0\\ 0 & 0 & \gamma\end{bmatrix}$, $M_1 \in M(3\times3)$ and $\langle\ ,\ \rangle$ denotes usual inner product.

Theorem 1. Let M be given as above.

(1) ξ is continuous

$\Longleftrightarrow$ $M_1 = M + p\,{}^{T}a$ where ${}^{T}a = (1\ 0\ 1)$

$\Longleftrightarrow$ $\xi(x) = Mx + \frac{1}{2} p \{|\langle a,x\rangle - 1| + (\langle a,x\rangle - 1)\}$

Hence, if ξ is continuous, M_1 is uniquely determined by p.

(2) If ξ is continuous, p is uniquely determined by eigenvalues $\lambda_1, \lambda_2, \lambda_3$ ($\in \mathbb{C}$) of M_1. Indeed,

$$p = \begin{bmatrix} 1 & 0 & 1 \\ \sigma & \omega & \gamma \\ \sigma^2-\omega^2 & 2\sigma\omega & \gamma^2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \begin{aligned} c_1 &= a_1 - b_1 \\ c_2 &= a_2 - b_2 + a_1 c_1 \\ c_3 &= a_3 - b_3 + a_1 c_2 + a_2 c_1 \end{aligned}$$

a_1 = trace M, $a_2 = -(2\sigma\gamma+\sigma^2+\omega^2)$, a_3 = det M,

b_1 = trace M_1, $b_2 = -(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)$, b_3 = det M_1.

<u>Remark 1.</u> Theorem 1 ensures that if we take any set of eigenvalues of M_1, the corresponding continuous piecewise-linear vector field exists uniquely.

<u>§ 2. Poincaré maps.</u>

Assume (1) ξ is continuous,

(2) the eigenvalues of M are $\sigma \pm i\omega$ ($\sigma > 0$, $\omega = 1$) and $\gamma < 0$, and

(3) the eigenvalues of M_1 are $\tilde{\sigma}_1 \pm i\tilde{\omega}_1$ ($\tilde{\sigma}_1 > 0$, $\tilde{\omega}_1 > 0$) and $\tilde{\gamma}_1 > 0$.

Set $\sigma_1 := \tilde{\sigma}_1/\tilde{\omega}_1$, $\gamma_1 := \tilde{\gamma}_1/\tilde{\omega}_1$ and $\mu := (\sigma, \gamma, \sigma_1, \gamma_1, \tilde{\omega}_1)$ (eigenvalue parameter).

<u>Definition 2.</u>

$V = \{x \in \mathbb{R}^3 : \langle a,x\rangle = 1\}$, $W = \{x=(x,y,z): x \le 0,\ y = 0,\ 0 \le z \le 1\}$

$L = \{x \in V : \langle a,\xi(x)\rangle = 0\}$, P = the singular point other then O.

$E^u(O), E^s(O)$ ($E^u(P)$, $E^s(P)$); (un)/stable eigen space at O (resp. P).

$A = E^u(O)\cap E^s(P)\cap V = {}^T(1, \sigma-(\sigma^2+1)/\tilde{\gamma}_1, 0)$, $\quad B = E^u(O)\cap L = {}^T(1,\sigma,0)$,

$C = E^u(P)\cap V$, $\quad D = E^s(O)\cap V$, $\quad E = E^s(P)\cap L$,

$F = \{x \in V : \xi(x)$ is parallel to $L\}$.

$\pi_1 : W \longrightarrow V$; $\pi_1(x) = \exp(Mt)x$, $\quad t = t(x) = \inf\{s>0 : \exp(Ms)x \in V\}$

$\pi_2 : V \longrightarrow V$; $\pi_2(x) = P + \exp(M_1 t)(x-P)$, $\quad t = t(x) = \inf\{s>0 : P+\exp(Ms)(x+P) \in V\}$

$\pi_3 : V \longrightarrow W$; $\pi_3(x) = \exp(Mt)x$, $\quad t = t(x) = \inf\{s>0 : \exp(Ms)x \in W\}$.

$A' = \pi_1^{-1}(A)$, $\qquad B' = \pi_1^{-1}(B)$.

Definition 3. Define a Poincaré map π of ξ as follows;

$\pi = \pi_{\mu} : W \longrightarrow W$

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$$\pi : \begin{cases} W \longrightarrow W;\ x \longrightarrow \exp(2\pi M)x & \text{if } x \in W \text{ does not hit } V \\ W \xrightarrow[\pi_1]{} V \xrightarrow[\pi_2]{} V \xrightarrow[\pi_3]{} W;\ x \longrightarrow \pi_3\pi_2\pi_1(x) & \text{if } x \in W \text{ hits } \Delta\ ABE \\ \pi_3(C) & \text{otherwise} \end{cases}$$

Theorem 2. π is differentiable on $\pi_1^{-1}(\overline{BF}) \cap W$.

Remark 2. Form Theorem 2, π is of class C^1 at all points of W except $\pi_1^{-1}(\overline{AE})$.

Definition 4. Let $F_n:[0,1]^2 \longrightarrow [0,1]^2$; $F_n(x,y) = (f_n(x,y), g_n(x,y))$ and $f:[0,1] \longrightarrow [0,1]$ be continuous (resp. of class C^1).

2D-maps F_n converges uniformly (resp. in C^1-sense) to 1D-map f as $n \longrightarrow \infty$ (denote $F_n \longrightarrow f$ (uniformly) (resp. C^1-sense))

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where $F:[0,1]^2 \longrightarrow [0,1]^2; F(x,y) = (f(x),0)$.

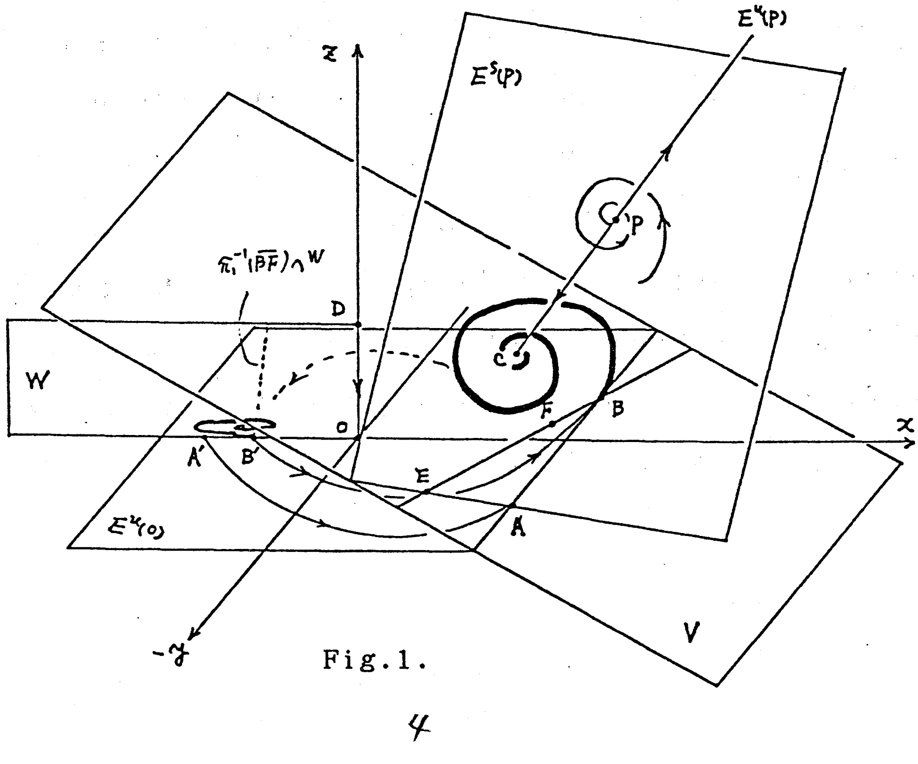

Fig.1.

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<u>Theorem 3.</u> Under the condition (*) for $\sigma, \tilde{\sigma}_1, \tilde{\omega}_1, \tilde{\gamma}_1$(see Remark 4 below),

(1) π converges uniformly to a 1D-map f as $\gamma \longrightarrow -\infty$.

(2) If $x \in [0,B']$, then $f(x) = \exp(2\pi\sigma)x$.

(3) There exist two explicit elementary functions $\varphi(t)$, $\psi(t)$ of $t \in [0,\infty)$ (see Remark 3) such that

$\forall x \in [B',A']$, $\exists t \in [0,\infty)$ s.t. $x = \varphi(t)$, $f(x) = \psi(t)$.

(4) $\lim_{x \to B'} f'(x) = \exp(2\pi\sigma)$.

<u>Remark 3.</u> Set

$A_0 = {}^{\top}(1, \sigma-(\sigma^2+1)/\tilde{\gamma}_1)$, $\quad B_0 = {}^{\top}(1,\sigma)$,

$A_1 = {}^{\top}(\gamma_1-2\sigma_1, 1-\sigma_1(\sigma_1-\gamma_1))\gamma_1/[(\sigma_1-\gamma_1)+1]$, $\quad B_1 = {}^{\top}(1,\sigma_1)$,

$$\alpha(t) = {}^{\top}(\exp(\sigma_1 t)\cdot\cos t - \exp(\gamma_1 t), -\exp(\sigma_1 t)\cdot\sin t) \in \mathbb{R}^2$$

$$u(t) = \{\langle\alpha(t),B_1\rangle + \exp(\gamma_1 t) - 1\}/\langle\alpha(t), B_1 - A_1\rangle, \quad v(t) = 1 - u(t),$$

$$x(t) = u(t)A_0+v(t)B_0 \in \mathbb{R}^2,$$

$$y(t)=\frac{\sigma^2+1}{\sigma_1^2+1}\begin{bmatrix} \gamma_1-\sigma_1 & 1 \\ (\gamma_1-\sigma_1)(\sigma_1-\gamma_1\sigma)-1 & \gamma_1(1-\sigma_1) \end{bmatrix}\{e^{\begin{bmatrix}\sigma_1 & -1 \\ 1 & \sigma_1\end{bmatrix}t}(u(t)B_1+v(t)A_1)-B_1\}+A_0 .$$

Then

$$\varphi(t)= |x(t)|\exp[-\sigma_1(\pi+\arg x(t))], \quad \psi(t)=|y(t)|\exp[\sigma_1(\pi-\arg y(t))].$$

<u>Remark 4.</u> The condition (*) of Theorem 3;

(1) $du/dt > 0$ for $\forall t \in [0,\infty)$, (2) $\{y(t):t>0\}\cap\{(x,y):y=0,x\le 0\} = \phi$.

For example, $\sigma=0.04$, $\tilde{\sigma}_1=-0.375$, $\tilde{\omega}_1=1.25$, $\tilde{\gamma}_1=1.0$ satisfy this condition.

<u>Conjectur.</u> π converges to the 1D-map f in C^1-sense as $\gamma \longrightarrow -\infty$.

<u>Definition 5.</u> Let $f:[0,1]\longrightarrow[0,1]$ be differentiable. An $x_0 \in [0,1]$ is stable (unstable) p-periodic point of f if

(i) $f^p(x_0) = x_0 \neq f^k(x_0)$ $(1 \le k < p)$,

(ii) $\left| \frac{df^p}{dx}(x_0) \right| < 1$ (> 1).

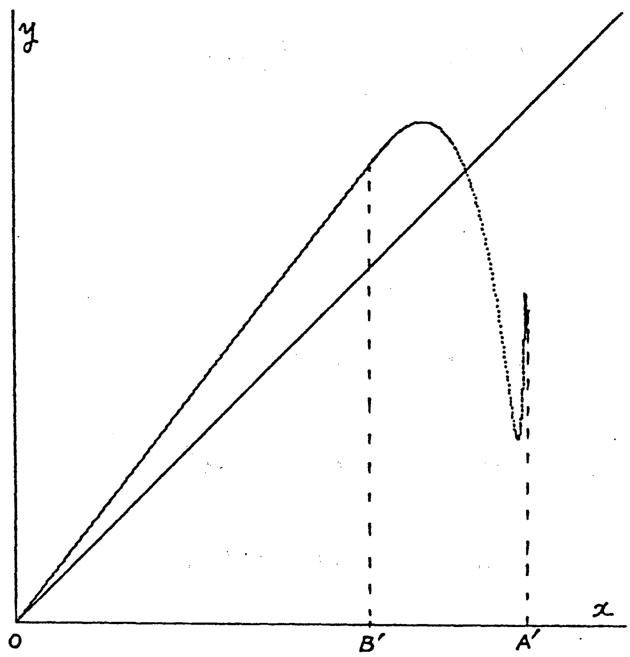

Fig.2. 1D-map ($\sigma=0.04,\omega=1,\tilde{\sigma}_1=-0.375,$ $\tilde{\omega}_1=1.25,\tilde{\gamma}_1=1$)

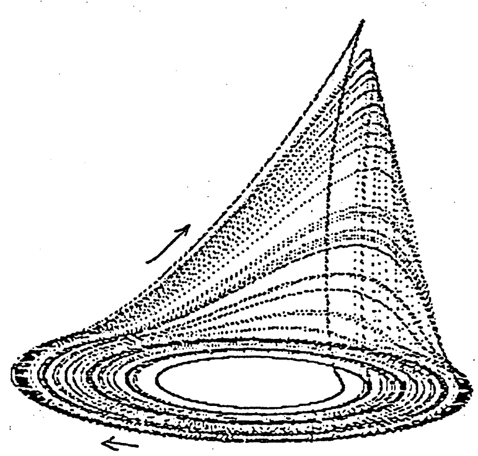

Fig.3. Spiral attractor in piecewise-linear vector fields ($\sigma=0.04,\omega=1,\tilde{\sigma}_1=-0.375,$ $\tilde{\omega}_1=1.25,\tilde{\gamma}_1=1$)

<u>Theorem 4.</u> Let $F_n:[0,1]^2 \longrightarrow [0,1]^2$ be continuous and $f:[0,1] \longrightarrow [0,1]$ differentiable. Assume $F_n \longrightarrow f$ (uniformly). If f has stable (resp. unstable) p-periodic point x_0, then

(i) $\exists N > 0$, $\forall n > N$, $\exists (x_0^n, y_0^n) \in [0,1]^2$ s.t.

$$F_n^p(x_0^n, y_0^n) = (x_0^n, y_0^n) \neq F_n^k(x_0^n, y_0^n) \quad (1 \leq k < p),$$

(ii) $(x_0^n, y_0^n) \longrightarrow (x_0, 0) \quad (n \longrightarrow \infty)$.

Moreover, if $F_n \longrightarrow f$ (C^1-sense), then (x_0^n, y_0^n) is stable (resp. saddle).

§ 3. Limit Systems.

<u>Theorem 5.</u> Set $\tilde{\xi}(x) = Q^{-1}\xi(Qx)$, where

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |\gamma| \end{bmatrix}.$$ As $\gamma \longrightarrow -\infty$, $\tilde{\xi}$ converges to the following system;

(1) $z > 0 \implies$

$$\begin{cases} \dot{x} = \sigma x - y + (b_2 + 2\sigma b_1 + 1 - 2\sigma^2)z \\ \dot{y} = x + \sigma y + (b_3 + \sigma b_2 + (\sigma^2 - 1)b_1 - 3\sigma(\sigma^2 - 1))z \\ \dot{z} = x - 1 \end{cases}$$

(where $b_i\,(1 \leq i \leq 3)$ are defined in Theorem 1(2)).

(2) $z < 0 \implies$

$$\begin{cases} \dot{x} = \sigma x - y \\ \dot{y} = x + \sigma y \\ \dot{z} = -\infty z \end{cases}$$

(3) $z = 0$ & $x \leq 1 \implies$

$$\begin{cases} \dot{x} = \sigma x - y \\ \dot{y} = x + \sigma y \\ \dot{z} = 0 \end{cases}$$

(4) $z = 0$ & $x > 1 \implies$

$$\begin{cases} \dot{x} = \sigma x - y \\ \dot{y} = x + \sigma y \\ \dot{z} = x - 1 \end{cases}$$

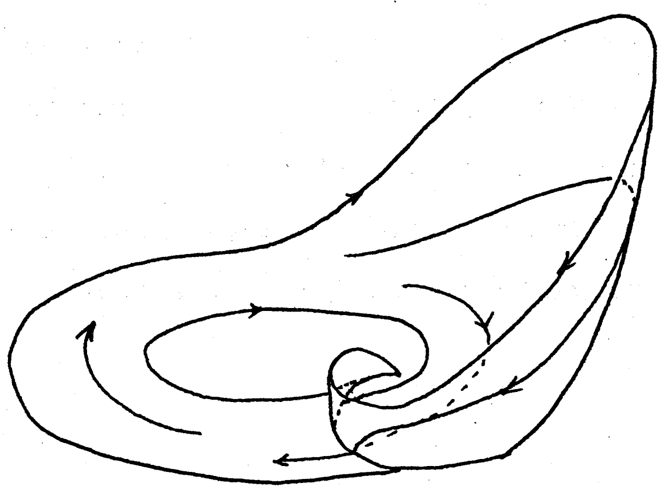

Fig.4.

<u>Remark 5.</u> These vector fields define a semi-flow on a 2-dimensional branched manifold as depicted in Figure 4, that is, the paper sheet model of the spiral attractor introduced by Rössler [5].

<u>References.</u>

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